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Power Laws and Multiplicative Processes

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Relationship between power laws and multiplicative processes which appear often in economics and finance are investigated. In particular, simple multiplicative factors and Kesten variables are explicitly studied. The master equations and Fokker-Planck equations are derived and utilized to show that the models acquire a power law in the asymptotic regions and that the Gibrat's law and detailed balance are necessary for the power law.

1. Introduction

Power laws,[1)~4)] characterized by their decreasing probability distributions in powers of variables over a wide range, have been extensively discussed in recent years in many areas such as economics, sociology, natural sciences, etc. For example, income distributions in many countries show in the higher range the power law called Pareto's law[5), 6)]

$$p(w) \sim w^{-1-\mu} \quad , \tag{1}$$

or in terms of cumulative distribution functions over the range greater than w

$$p_{>}(w) \sim w^{-\mu} \quad , \tag{2}$$

where w is the income and the exponent μ is called Pareto exponent. The exponent is either greater than one or almost equal to one in most cases. Although Pareto proposed this law a century ago, the validity has been established very recently in many countries largely due to intensive studies initiated by scientists in econophysics. Other economic examples are size distributions of firms,[7)] cities,[8)] stock returns,[1), 9)] rate distributions of firm growth,[10)] etc. This law is observed in other fields as well.[4)]

When a slab of plaster is crashed, the size distribution of the broken pieces obeys the law.[11)] The earthquake magnitudes follow it, too.[2)] The geographical characters such as moon craters belong to the same law.[12)] Word frequency in a book also shows the law with the exponent of almost one and is now called Zipf's law.[13)] The network analysis of the Internet, transport and social relations indicates that the degree distribution which measures the number of links from each node observes the similar law.[3), 14)] Thus, the power laws are ubiquitous in our world.

The power law implies scale invariance that if we magnify the size distribution, we see the identical pattern of the distribution. If the variable is dilated by a factor of L,

$$w \to Lw \, , \tag{3}$$

then, the distribution is transformed as

$$p(w) \to p'(w) = p(Lw) = L^{-1-\mu} p(w) \, . \tag{4}$$

The new distribution, however, looks the same as before since

$$L^{1+\mu} p'(w) = p(w) \, . \tag{5}$$

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When we observe stock price changes as a function of time, we see a violent pattern. This pattern appears the same when we enlarge the data along the time axis. That is, the law indicates the loss of standard measure. This state of affairs is connected to the fractals discovered by Mandelbrot.[11]

The ubiquity of the power law is puzzling especially in social fields because humans can take into account past experiences in their decisions so that the outcome after each decision must be different. The realities, however, imply that the distributions are rather stable and demand an explanation.

As early as 1931 Gibrat claimed that the firm size increases with a uniform rate across various firms irrespective of the size of each firm.[15] The situation can be compared with a compound interest rate which always favors larger deposits because it produces ever greater amounts for them. When we introduce the simple multiplicative process, however, we end up with the log-normal distribution[15] and does not observe a steady state. Levy and Solomon, then, proposed a boundary condition that the size does not become smaller than a critical value.[16] This seems reasonable because a personal income, for example, can not certainly go lower than a minimum value in order to live. The boundary constraint allows for a steady solution which acquires an exponential factor much like the Boltzmann factor when the problem is transformed to the random walk. Sornette and Cont interpreted that the boundary condition acts as a wall to drive back the walker who approaches it with a negative velocity.[17] Were it not for this wall, the walker would travel in the remote distance and correspondingly the probability distribution would vanish everywhere. They further suggested that the power law should emerge if there is another mechanism to keep the walker from escaping to the infinity. In fact, the Kesten process[18, 19] which adds an extra random potential to the term with the simple multiplicative factor reveals the power law in a steady state. They argue that the random potential acts as a wall. The Kesten model was investigated further by Takayasu et al.[20] and by Sornette[21] and the power law was confirmed. Actually, the existence of the power law has been rigorously proven by Kesten[18] and Goldie.[19] It is now called Kesten-Goldie theorem. Later, investigating the firm size distribution, Fujiwara et al. discovered that the detailed balance condition combined with the Gibrat's law results in the power law.[22] This is understandable because the detailed balance condition guarantees the existence of a time-independent solution. Gabaix derived the condition of Pareto exponent to be one in connection with the city evolution by assuming that the Gibrat's law is valid only in the asymptotic regions.[23]

In the present paper we clarify the relationship between the power law and the multiplicative processes by explicitly investigating two models, a simple multiplicative process and the Kesten process. We derive the master equations first and then, in the continuum approximation the Fokker-Planck equations for each model. In the asymptotic regions we confirm the power law as well as the condition obtained by Gabaix.[23]

2. Models

(1) Simple multiplicative process

This process represents a random multiplicative influence for the amount of an individual income or the size of a city as

$$w_{t+1} = \lambda_t w_t \,, \tag{6}$$

where w_t indicates the size of a city or an individual income and $\lambda_t > 0$ with the distribution $\Pi(\lambda)$. The probability density of w is defined by

$$P_{t+1}(w) = \langle \delta(w - w_{t+1}) \rangle_\lambda . \qquad (7)$$

Taking average over λ, we obtain the master equation

$$P_{t+1}(w) = \int d\lambda \Pi(\lambda) \frac{1}{\lambda} P_t\left(\frac{w}{\lambda}\right) \cdot \qquad (8)$$

Factor $1/\lambda$ is missing from the master equation in Levy and Solomon.[16] A similar equation for the accumulated distribution $P_{>t+1}(w)$ holds but without the factor $1/\lambda$ inside the integrand. We can derive the Fokker-Planck equation by expanding the distribution function in terms of the small time interval τ around $\lambda = 1$,

$$\dot{P}(w,t) = \frac{1}{\tau}\left[M_0 - 1 + (-M_1 + M_2) w \frac{\partial}{\partial w} + \frac{1}{2} M_2 w^2 \frac{\partial^2}{\partial w^2} \right] P(w,t) , \qquad (9)$$

where t the continuous time and

$$\begin{aligned} M_0 &= \int d\lambda \Pi(\lambda) \frac{1}{\lambda} \\ M_1 &= \int d\lambda \Pi(\lambda) \frac{1}{\lambda}(\lambda - 1) = 1 - M_0 \\ M_2 &= \int d\lambda \Pi(\lambda) \frac{1}{\lambda}(\lambda - 1)^2 = \bar{\lambda} - 2 + M_0 \end{aligned} \qquad (10)$$

with $\bar{\lambda}$, the average value over Π.

Eq. (6) is easier to handle by new variables defined by

$$\begin{aligned} x_t &= \ln w_t \\ l_t &= \ln \lambda_t \end{aligned} \qquad (11)$$

because the equation becomes the standard random walk

$$x_{t+1} = x_t + l_t , \qquad (12)$$

where l_t obeys the distribution $\pi(l)$. The new distribution functions are related by the old ones

$$\begin{aligned} \pi(l) &= \lambda \Pi(\lambda) \\ p_t(x) &= w P_t(w) \end{aligned} . \qquad (13)$$

In terms of the transformed variables x and l, the master equation is given by

$$p_{t+1}(x) = \langle \delta(x - x_t) \rangle_l . \qquad (14)$$

Taking average over l, we find

$$p_{t+1}(x) = \int dl \pi(l) p_t(x - l) . \qquad (15)$$

Expanding the distribution function in powers of τ and l, we obtain the Fokker-Planck equation

$$\dot{p}(x,t) = -\left(v \frac{\partial}{\partial x} - D \frac{\partial^2}{\partial x^2} \right) p(x,t), \qquad (16)$$

where the average velocity and the diffusion coefficient are given by

$$v = \frac{\bar{l}}{\tau} \qquad (17)$$

and

$$D = \frac{\overline{(l - \bar{l})^2}}{2\tau} , \qquad (18)$$

respectively and

$$\bar{l} = \int dl \pi(l) l = \int d\lambda \Pi(\lambda) \ln \lambda . \qquad (19)$$

The coefficients M_i ($i=0, 1, 2$) can be expressed in terms of v and D so that the Fokker-Planck equation for $P(w, t)$ is now

$$\dot{P}(w,t) = -\left[v - D + (v-3D)w\frac{\partial}{\partial w} - Dw^2\frac{\partial^2}{\partial w^2} \right] P(w,t) . \tag{20}$$

We can solve this equation simply by resorting to the equivalent equation in Eq. (16)

$$P(w,t) = \frac{1}{\sqrt{2\pi Dt}}\frac{1}{w}\exp\left[-\frac{1}{2Dt}(\ln w - vt)^2\right], \tag{21}$$

which is the so-called log-normal distribution, the Gibrat's result.[15] In the limit $t \rightarrow \infty$, $P \rightarrow 0$ so that a finite solution does not exist for Eq. (20). However, as was pointed by Levy and Solomon,[16] if we introduce the boundary condition $w > w_0$, we obtain the finite solution for $v < 0$

$$P(w) = \mu w_0{}^{\mu} w^{-1-\mu} , \tag{22}$$

where

$$\mu = \frac{|v|}{D} . \tag{23}$$

The reason of the existence of the stationary solution for $v < 0$ lies in the boundary constraint which repels a random walker moving in the left direction like a wall as discussed by Sornette and Cont.[17]

Alternatively, the exponent can be determined directly by the master equation. In the steady state Eq. (8) yields

$$P(w) = \int d\lambda \Pi(\lambda)\frac{1}{\lambda}P\left(\frac{w}{\lambda}\right), \tag{24}$$

or à la Gabaix[23]

$$P_>(w) = \int d\lambda \Pi(\lambda) P_>\left(\frac{w}{\lambda}\right). \tag{25}$$

Therefore, we can determine the exponent from either equation, assuming in the large w limit,

$$P(w) \approx C w^{-1-\mu} . \tag{26}$$

The exponent μ satisfies

$$1 = \int d\lambda \Pi(\lambda)\lambda^{\mu} . \tag{27}$$

In particular the Pareto exponent equals one when

$$\overline{\lambda} = 1 . \tag{28}$$

Note that the exponent does not depend on the constraint w_0 and that the boundary condition is necessary only to guarantee the existence of a finite stationary solution.

(2) Kesten process

The Kesten process is an extension of the simple multiplicative process discussed in (1) by an additional random potential which can be thought of as an external stimulus for growth such as immigration into a city.[21] The model[18), 19)] is defined by

$$s_{t+1} = \lambda_t s_t + b_t , \tag{29}$$

where $\lambda_t > 0$, $b_t > 0$ with the distribution $\Pi_b(b_t)$. The probability density function is defined by

$$P_{t+1}(s) = \langle \delta(s - s_{t+1}) \rangle_{\lambda, b} . \tag{30}$$

Performing the averages, we find the master equation

$$P_{t+1}(s) = \int d\lambda db \Pi(\lambda)\Pi_b(b)\frac{1}{\lambda}P_t\left(\frac{s-b}{\lambda}\right). \quad (31)$$

A similar equation for the accumulated distribution $P_{t+1>}(s)$ holds but without the factor $1/\lambda$ inside the integrand. In the limit $s \gg 1$

$$P_{t+1}(s) = \int d\lambda \Pi(\lambda)\frac{1}{\lambda}\left[1 - \bar{b}\frac{\partial}{\partial s} + \frac{1}{2}\overline{b^2}\frac{\partial^2}{\partial s^2} + ...\right]P_t\left(\frac{s}{\lambda}\right) \cdot (32)$$

In the steady state, then,

$$P(s) = \int d\lambda \Pi(\lambda)\frac{1}{\lambda}\left[1 - \bar{b}\frac{\partial}{\partial s} + \frac{1}{2}\overline{b^2}\frac{\partial^2}{\partial s^2} + ...\right]P\left(\frac{s}{\lambda}\right) \cdot (33)$$

or

$$P_>(s) = \int d\lambda \Pi(\lambda)\left[1 - \bar{b}\frac{\partial}{\partial s} + \frac{1}{2}\overline{b^2}\frac{\partial^2}{\partial s^2} + ...\right]P_>\left(\frac{s}{\lambda}\right). \quad (34)$$

If the distribution $P(s)$ decays as the power law

$$P(s) \approx cs^{-1-\mu}, \quad (35)$$

then, the higher order terms of Eqs. (34) and (33) inside the bracket of each integrand can be ignored and the Gibrat's law holds

$$P(s) = \int d\lambda \Pi(\lambda)\frac{1}{\lambda}P\left(\frac{s}{\lambda}\right), \quad (36)$$

or

$$P_>(s) = \int d\lambda \Pi(\lambda)P_>\left(\frac{s}{\lambda}\right). \quad (37)$$

Both equations give rise to

$$1 = \int d\lambda \Pi(\lambda)\lambda^{\mu}, \quad (38)$$

which is called the Kesten-Goldie theorem. The theorem has earlier been proven by the renewal theory.[18), 19)] The Pareto exponent equals one when Eq. (28) holds as before. Note that the Pareto exponent is independent of the additional potential b in Eq. (29). Now, we check whether the master equation in Eq. (31) has a stationary solution or not.

The Fokker-Planck equation can be obtained as in Eq. (9)

$$\begin{aligned}\dot{P}(s, t+1) = -\Bigg[& v - D + (v - 3D)s\frac{\partial}{\partial s} \\ & - Ds^2\frac{\partial^2}{\partial s^2} + v_b\frac{\partial}{\partial s} - D_b\frac{\partial^2}{\partial s^2}\Bigg]P(s, t).\end{aligned} \quad (39)$$

where

$$v_b = \frac{\bar{b}}{\tau}$$
$$D_b = \frac{\overline{\left(b - \bar{b}\right)^2}}{2\tau}. \quad (40)$$

The first four terms inside the bracket on the right hand side correspond to the multiplication process represented by λ as in Eq. (20) and the last two terms come from the additive fluctuation term associated with b. In terms of new variables

$$z_t = \ln s_t \quad (41)$$

the Fokker-Planck equation reduces to

$$\begin{aligned}\dot{p}(z, t) = -\Bigg[& v\frac{\partial}{\partial z} - D\frac{\partial^2}{\partial z^2} - v_b e^{-z}\left(1 - \frac{\partial}{\partial z}\right) \\ & - D_b e^{-2z}\left(2 - 3\frac{\partial}{\partial z} + \frac{\partial^2}{\partial z^2}\right)\Bigg]p(z, t).\end{aligned} \quad (42)$$

In the limit $z \gg 1$ the terms of exponential factors drop out and

$$\dot{p}(z, t) = -\left[v\frac{\partial}{\partial z} - D\frac{\partial^2}{\partial z^2}\right]p(z, t), \quad (43)$$

which is the random walk treated in Eq. (16). In the opposite limit $z \ll -1$, on the other hand, the exponential terms dominate

$$\dot{p}(z,t) = -\left[-v_b e^{-z}\left(1-\frac{\partial}{\partial z}\right) - D_b e^{-2z} \left(2 - 3\frac{\partial}{\partial z} + \frac{\partial^2}{\partial z^2}\right)\right] p(z,t) \, . \tag{44}$$

These equations give rise to asymptotic behaviors in the steady state for $v < 0$

$$p(z) \approx \begin{cases} c_1 e^{-\mu z} & \text{for } z \gg 1 \\ 0 & \text{for } z \ll -1 \end{cases} \, . \tag{45}$$

The distribution shows the power law at the higher end in the original variable due to Eq. (41).

Note that the Pareto exponent does not depend on the additional potential b in Eq. (29) and that the additional potential operates to ensure the existence of a stationary solution as in (1).

3. Conclusions

In this paper we have derived the master equation and the Fokker-Planck equations for the simple multiplicative process and the Kesten process.[19),18)] We have shown explicitly that in the asymptotic region the master equations can be reduced to the equation for the Pareto exponent μ which agrees with the previous studies.[23),18),19),17),16)] In particular, the exponent equals one when the average of the multiplicative factor λ is one as Gabaix[23)] claimed. Furthermore, we have shown explicitly this power law in the asymptotic regions by solving the Fokker-Planck equations. We conclude that in both processes in (1) and (2) the Gibrat's law is necessary but not sufficient for the power law and that either the constraint in the simple multiplicative process or the additional potential in the Kesten process are required in order to support a stationary state and produce the power law.

Currently, macroeconomics is expanded into dynamical processes in order to explain the growth or cyclic mechanism of the economy. Until now, however, major efforts have been directed towards incorporating the microeconomics into the macroeconomics by considering a representative agent. In reality macro economy involves many heterogeneous agents who are interacting among themselves. Stochastic processes which we have discussed in this paper are just one of approaches to account for this heterogeneity. An evolutionary approach[25),24)] is also promising. The other method resorts to the combinatorial stochastic processes where distribution of agents into clusters are studied. This method, too, can yield the power law as in Ewens sampling formula and is explained by Aoki and Yoshikawa.[26)] They claim that neoclassical macroeconomics which ignores the heterogeneity is insufficient and Keynesian approach is fundamentally correct. Furthermore, they seem to believe that a micro economic foundation for macroeconomics is impossible. Scientific theories, however, do exist where we succeed in deriving macroscopic equations from microscopic theories. This then should open a new development for macroeconomics in the future.

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